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Variable-Coefficient A-Stable Explicit Runge-Kutta Methods

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Abstract

We study the explicit Runge-Kutta methods for stiff-equation $y' = \lambda y$,where the methods are variable coefficients formulas depending on λ . They are A-stable with respect to the model equation $y' = \lambda y$. The analysis of eigenvalues λ for some schemes are carried out .

1. Introduction

The present paper is concerned with the numerical integration of stiff system of ordinary differential equation:

$$y' = f(x,y), \quad y(x_0) = y_0. \tag{1.1}$$

A basic difficulty in the numerical solution of stiff system is the satisfying of the requirement of stability, from the restriction of stability, implicit type methods have been present and some explicit methods imposed the stability conditions have derived, however, there still remain stability problem for the explicit methods, so it is the purpose of the present paper to derive the explicit A-stable Runge-Kutta methods with respect to the model equation. The outline of this paper is as follows: In § 2, We consider two-stage of order one, three-stage of order two and four-stage of order three explicit A-stable Runge-Kutta methods for the fitting problem respectively. Stability analysis for arbitrary eigenvalue λ are discussed in § 3 . In § 4, we proposed some numerical tests.

2. Derivation of the formulae

Consider the r-stage explicit Runge-Kutta methods:

$$y_{n+1} = y_n + h \sum_{i=1}^{r} b_i k_i \quad , \tag{2.1}$$

$$k_1 = f(x \ , y \ ),$$

$$k_i = f(x \ + c_i h, y \ + h \ \Sigma \ a_{ij} k_j),$$

$$c_i = \Sigma \ a_{ij} \quad (i = 2,..,r).$$

The order conditions of the R-K methods which are discussed in [1], are listing up to three order:

order 1: $\Sigma b_i = 1,$ (2.2)

order 2: $\Sigma \ b_i c_i = 1/2,$ (2.3)

order 3: $\Sigma \ b_i c_i^2 = 1/3,$ (2.4)

$\Sigma b_i a_{ij} c_j = 1/6.$

Let us now apply the r-stage, p-th order Runge-Kutta methods (2.1) to the test equation

$$y' = \lambda \ y \tag{2.5}$$

then we have

$$y_{n+1} = S(z) y_n \ , \tag{2.6}$$

and $S(z)$ takes the form

$$S(z) = \sum_{i=1}^{p} \frac{z^i}{i!} + \sum_{\kappa=p+1}^{r} \gamma z^{\kappa}, \quad (z = \lambda h)$$

where γ are the function of the coefficients of (2.1).

We shall study how the function $S(z)$ of (2.6) with $(p,r) = (1,2),(2,3)$ and $(3,4)$ are expressed.

Case (1) $p=1,r=2$: From (2.6) we obtain the difference equation:

$$y_{n+1} = (1 + z + b_2 a_{21} z^2)y_n, \tag{2.7}$$

here if we take $b_2 a_{21}$ in the form:

$$b_2 a_{21} = \frac{\delta}{\alpha + \beta z}, \tag{2.8}$$

then from (2.7) and (2.8) we have

$$y_{n+1} = \frac{\alpha + (\alpha+\beta)z + (\beta + \delta)z^2}{\alpha + \beta z} y_n. \tag{2.9}$$

From the stability condition, we have $\beta + \delta = 0$, taking, for example, $\alpha = 1$, $\beta = -1$ we have

$$y_{n+1} = \frac{1}{1 - z} y_n, \tag{2.10}$$

which is A-stable algorithm. Solving (2.8) and the order condition (2.2), we have

$$b_2 = \frac{1}{c_2(1 - z)}, \quad b_1 = 1 - b_2, \tag{2.11}$$

$(c_2$:free parameter)

Case(11) $p=2,r=3$: Proceeding the same way as the case (1), we have

$$y_{n+1} = (1 + z + \frac{z^2}{2!} + b_3 a_2 a_{31} z^3_{21})y_n, \tag{2.12}$$

setting

$$b_4 a_{32} a_{21} = \frac{\gamma}{2!(\alpha + \beta z)}, \tag{2.13}$$

we have

$$y_{n+1} = \frac{2\alpha + 2(\alpha+\beta)z + (2\beta+\alpha)z^2 + (\beta+\gamma)z^3}{3!(\alpha+\beta z)}\, y_n \,.$$

From the stability condition, we have

$$2\beta + \alpha = 0,\ \beta + \gamma = 0,$$

which lead to the following A-stable algorithm:

$$y_{n+1} = \frac{2+z}{2-z}\, y_n \,. \tag{2.14}$$

Solving (2.13) and order conditions, we have

$$a_{21} = c_2,\ a_{31} = c_3 - a_{32}\,,$$

$$b_3 = -\frac{1}{a_{32}a_{21}(2+\beta z)},\quad b_2 = \frac{1}{c_2}\left(\frac{1}{2} - b_3c_3\right), \tag{2.15}$$

$$b_1 = 1 - (b_2 + b_3). \quad (a_{32}:\text{free parameter})$$

Case (III) p=3,r=4: Finally in this section, we concern the four-stage three order method, integrating (2.5), we have

$$y_{n+1} = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + b_4\, a_{43}a_{32}a_{21}z^4\right)y_n \,, \tag{2.16}$$

here we set

$$b_4\, a_{43}a_{32}a_{21} = \frac{\gamma}{3!(\alpha+\beta z)}\,, \tag{2.17}$$

putting (2.17) into (2.16), we have

$$y_{n+1} = \frac{6\alpha + 6(\alpha+\beta)z + (6\beta+3\alpha)z^2 + (3\beta+\alpha)z^3 + (\beta+\gamma)z^4}{3!(\alpha+\beta z)}\, y_n \,.$$

From the stability condition, we have

$$6\beta + 3\alpha = 0,\ 3\beta + \alpha = 0,\ \beta + \gamma = 0,$$

which lead to

$$\alpha = \beta = 0 .$$

It follows that the assupution (2.17) is unsuitable. We now consider the following further case:

$$b_4\, a_{43}a_{32}a_{21} = \frac{\delta + \rho z}{\alpha + \beta z + \gamma z^2} , \tag{2.18}$$

putting (2.18) into (2.16), we have

$$y_{n+1} = \frac{u}{3!(\alpha + \beta h + \ z^2)}\, y_n ,$$

with

$$u = 6\alpha + 6(\alpha + \beta)z + (6\gamma + 6\beta + 3\alpha)z^2 + (6\gamma + 3\beta + \alpha)z^3$$
$$+ (3\gamma + \beta + 6\delta)z^4 + (\gamma + 6\rho)z^5 .$$

From the stability condition we have

$$\gamma + 6\rho = 0,\ 3\gamma + \beta + 6\delta = 0,\ 6\gamma + 3\beta + \alpha = 0,$$

which lead to

$$\alpha = -9\delta,\ \beta = 11\delta,\ \gamma = -4\delta,\ \rho = 4\delta . \tag{2.19}$$

If we take $\delta = 1$ in (2.19), we have the following L-stable algorithm:

$$y_{n+1} = \frac{-9 + 2z}{6(9 - 11z + 4z^2)}\, y_n . \tag{2.20}$$

Solving (2.18) and order conditions (2.2),(2.3),(2.4), we have

$$b_4 = \frac{1}{a_{43}a_{32}a_{21}}\ \frac{1 + 4z}{(-9 + 11z - 4z^2)}, \tag{2.21}$$

$$b_2 = \frac{1}{c_2(c_3 - c_2)}\left\{\frac{1}{2} - b_4c_4)c_3 - \frac{1}{3} + b_4c_4^2\right\},$$

$$b_3 = \frac{1}{c_3}\left\{\frac{1}{2} - b_4c_4 - b_2c_2\right\},$$

$$a_{42} = \frac{1}{b_4 c_2}\left\{\frac{1}{6} - b_3 a_{32} c_2\right\} - \frac{c_3}{c_2} a_{43} ,$$

$$a_{31} = c_2 , \quad c_3 = b_{31} + b_{32}, \quad a_4 = c_4 - (a_{42} + a_{43}).$$

$(a_{32}, a_{31}, b_{43}, c_2, c_4$:free parameters).

3. Stability properties of the schemes (2.10),(2.14) and (2.20).

In this section we are concerned with the analysis of eigenvalue of schemes (2.10),(2.14) and (2.20). Let us set λ' be the approximation of λ , replacing z by $z' = \lambda' h$ in (2.8), the algorithm (2.10) is then

$$y_{n+1} = (1 + z + \frac{z^2}{1 - z'}) y_n ,$$

or

$$y_{n+1} = \pi(z, z') y_n ,$$

$$\pi(z,z') = 1 + z + \frac{z^2}{1 - z'}, \tag{3.1}$$

we may write (3.1) in the form

$$\pi(z,z') = \left(\frac{1}{1-z} \quad \frac{z^2}{1-z'} - \frac{z^2}{1-z}\right) , \tag{3.2}$$

so if z' satisfies the equation

$$\left| z^2 \left(\frac{1}{1-z'} - \frac{1}{1-z}\right)\right| < \left|1 - \frac{1}{1-z}\right| , \tag{3.3}$$

then the algorithm (3.1) is A-stable. Setting $z = re^{i\theta}$ $(\pi/2 < \theta < 3\pi/2)$, and using the inequality

$$|1 - z| < 1 + r ,$$

in (3.3), we have

$$\left|\frac{z' - z}{z' - 1}\right| < \frac{1}{r + 2} ,$$

which lead to the following result.

Theorem 1. The the algorithm (2.11) with $z = z'$, $\alpha = 1$, $\beta = -1$ is A-stable if z' satisfies the inequality:

$$\left|\frac{z' - z}{z' - 1}\right| < \frac{1}{r + 2} . \tag{3.4}$$

The region z' satisfying (3.4) lies in the interior of the circle with the center m_1 and the radius r_1 ,

$$m_1 = -\frac{c^2 - z}{1 - c^2} , \quad r_1 = c \frac{1 - z}{1 - c^2} , \tag{3.5}$$

with $c = 1/(r+2)$. Taking the value of r large enough, we have the following result:

Corollary. For large value of z' in (2.11), the algorithm (2.11) with $\alpha = 1$, $\beta = -1$ is A-stabe if z' satisfies the inequality:

$$\left| z - z' \right| < 1 .$$

Carring the same argument to the algorithm (2.14) and (2.20), we have the following results:

Theorem 2. The algorithm (2.15) with $\gamma = -\beta$, $z = z'$ is A-stable if z' satisfies the inequality:

$$\left|\frac{z' - z}{z' - 2}\right| < \frac{1}{\gamma^3} (\sqrt{z_1} - \sqrt{z_2}) . \tag{3.6}$$

with

$$z_1 = r^2 - 4r\cos(\theta) + 4 ,$$

$$z_2 = r^2 + 4r\cos(\theta) + 4 ,$$

The region z' satisfing (3.6) is in the interior of the circle with the center m_2 and the radius r

$$m_2 = -\frac{c^2 - z^2}{1 - c^2}, \quad r_2 = c\,\frac{1 - z}{1 - c^2} \tag{3.7}$$

with $c_2 = 1/(r(r+2))$.

Theorem 3. The algorithm (2.21) with $\alpha = 6$, $\beta = -4$, $\gamma = 1$, $\rho = -1$, $\delta = 1$ and $z = z'$ is A-stabel if z' satisfies the following inequality:

$$\frac{|(z - z')(88zz' - 36(z + z') + 81|}{|(9 - 11z' + 4z'^2)(9 - 11z + 4z^2)|} < \frac{1}{|z^4|}\left\{1 - \frac{|-9 + 2z|}{|9 - 11z + z^2|}\right\}.$$

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